TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 330, Number 2, April 1992

A DEFORMATION OF TORI WITH CONSTANT MEAN CURVATURE IN $\mathbb{R}^3$ TO THOSE IN OTHER SPACE FORMS

MASAAKI UMEHARA AND KOTARO YAMADA

ABSTRACT. It is shown that tori with constant mean curvature in $\mathbb{R}^3$ constructed by Wente [7] can be deformed to tori with constant mean curvature in the hyperbolic 3-space or the 3-sphere.

INTRODUCTION

In this paper, we will construct tori with constant mean curvature in the hyperbolic 3-space. To be more precious, let T^2 be a torus and $f : T^2 \to \mathbb{R}^3$ be an immersion with constant mean curvature constructed by Wente [7]. Let

$$\mathbb{R}^3(k) = \begin{cases} \mathbb{R}^3 & (\text{if } k \geq 0), \\ \{x \in \mathbb{R}^3 : \sum_{i=1}^3 (x^i)^2 < \frac{1}{|k|}\} & (\text{if } k < 0) \end{cases}$$

be the Riemannian 3-manifold with the Riemannian metric

$$g_k = \left(\frac{2}{1 + k \sum_{i=1}^3 (x^i)^2} \right)^2 \sum_{i=1}^3 (dx^i)^2$$

of constant sectional curvature k. We will show that if f is generic, then for a sufficiently small $\varepsilon > 0$ there exists a local 1-parameter family of immersions $\{f_k : T^2 \to \mathbb{R}^3(k)\}_{|k|<\varepsilon}$ $(f_0 = f)$ with the same constant mean curvature. It should be noted that the induced metrics $\{f_k^* g_k\}_{|k|<\varepsilon}$ on T^2 in this case may not be conformally equivalent to each other. Recently Walter [6] gave another construction of tori with constant mean curvature in the hyperbolic 3-space. But our construction is quite different and depends very much on an idea "deformation of Lie groups".

Wente's construction in [7] is based on doubly periodic solutions of the sinh-Gordon equation on $\mathbb{R}^2$. Even if $k \neq 0$, solutions of the sinh-Gordon give rise to immersions $f_k : \mathbb{R}^2 \to \mathbb{R}^3(k)$ with constant mean curvature. Though f_k may not be doubly periodic, it induces a representation $\rho_k : \pi_1(T^2) \to G_k$ such

---

Received by the editors October 10, 1989 and, in revised form, May 4, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53A10; Secondary 53C42.
Key words and phrases. Surfaces with constant mean curvature.
The first author was partly supported by the Yukawa Foundation and University of Tsukuba Project Research.

that for any $a \in \pi_1(T^2)$, $\rho_k(a)$ preserves the image of f_k, where

$$G_k = \begin{cases} SO(4) & (\text{if } k > 0), \\ SO(3) \ltimes \mathbb{R}^3 & (\text{if } k = 0), \\ SO^+(3, 1) & (\text{if } k < 0), \end{cases}$$

which are the identity components of the isometry groups of the 3-dimensional space forms. The necessary and sufficient condition for the image of f_k to be closed can be described in terms of the representation ρ_k. To construct a family of doubly periodic immersions, one difficulty arises from the fact that the isometry groups G_k for $k > 0$, $k = 0$, and $k > 0$ are quite different from each other.

In §§1–3, we introduce a differentiable structure on the set $I = \{(k, E) : k \in \mathbb{R}, E \in G_k\}$ such that the family of representations $\rho_k : \pi_1(T^2) \to G_k \subset I$ $(k \in \mathbb{R})$ is smooth with respect to k. In §4, a criterion for the image of f_k to be closed can be taken depending smoothly on k, by virtue of the differentiable structure. Using this criterion, the existence of a deformation f_k with the desired properties are shown in the last section.

The authors would like to thank all those who encouraged them and gave them suggestions, and in particular Professors H. Ozeki and S. Nishikawa.

1. Decompositions of isometries

Let $M^3(k)$ be a complete simply connected Riemannian 3-manifold of constant sectional curvature k, and G_k the identity component of the isometry group of $M^3(k)$.

First, we suppose $k > 0$. In this case, $M^3(k)$ is the Euclidean sphere defined by

$$M^3(k) = \left\{ {}^t(x^1, x^2, x^3, t) \in \mathbb{R}^4 : \sum_{i=1}^{3} (x^i)^2 + t^2 = \frac{1}{k} \right\} \tag{1.1}$$

and $G_k = SO(4)$. It is well known that for each $E \in SO(4)$, there exists a matrix $P \in SO(4)$ such that

$$P^{-1} \circ E \circ P = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\nu & -\sin\nu \\ 0 & 0 & \sin\nu & \cos\nu \end{pmatrix}, \tag{1.2}$$

where $e^{\pm i\theta}$ and $e^{\pm i\nu}$ are the eigenvalues of the matrix E.

Next we consider the case $k < 0$. In this case, $M^3(k)$ is the hyperboloid in the Minkowski 4-space $\mathbb{L}^4$ with the induced metric. That is,

$$M^3(k) = \left\{ {}^t(x^1, x^2, x^3, t) \in \mathbb{L}^4 : \sum_{i=1}^{3} (x^i)^2 - t^2 = \frac{1}{k},\ t > 0 \right\} \tag{1.3}$$

and $G_k = SO^+(3, 1)$. Unlike the case $SO(4)$, not all matrices in $SO^+(3, 1)$ can be normalized.

Lemma 1.1. *Let*

$$N = \{ A \in SO^+(3, 1) : \textit{all of the eigenvalues of } A \textit{ are } 1 \}. \tag{1.4}$$

Then for any matrix $E \in SO^+(3,1)\backslash N$, *there exists* $P \in SO^+(3,1)$ *such that*

$$P^{-1} \circ E \circ P = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cosh\nu & \sinh\nu \\ 0 & 0 & \sinh\nu & \cosh\nu \end{pmatrix}, \tag{1.5}$$

where $e^{\pm i\theta}$ *and* $e^{\pm\nu}$ *are the eigenvalues of the matrix* E.

Proof. Identify a point $X = {}^t(x^1, x^2, x^3, t) \in \mathbb{L}^4$ with a 2×2-matrix

$$X = \begin{pmatrix} x^3 + t & x^1 + ix^2 \\ x^1 - ix^2 & -x^3 + t \end{pmatrix}.$$

Then $SO(3,1)$ is isomorphic to $PSL(2,\mathbb{C})$ by the 2-fold covering ρ: $SL(2,\mathbb{C}) \to SO^+(3,1)$ defined by $\rho(a)X = a\circ X \circ {}^t\bar{a}$. It is easy to check that

$$\rho\begin{pmatrix} e^{z/2} & 0 \\ 0 & e^{-z/2} \end{pmatrix} = \begin{pmatrix} \cos u & -\sin u & 0 & 0 \\ \sin u & \cos u & 0 & 0 \\ 0 & 0 & \cosh v & \sinh v \\ 0 & 0 & \sinh v & \cosh v \end{pmatrix}, \tag{1.6}$$

where $z = u + iv$. On the other hand, $\rho^{-1}(N) \subset SL(2,\mathbb{C})$ consists exactly of matrices which cannot be diagonalized. Combining these two facts, we obtain the lemma. □

Finally we consider the case $k = 0$. The following lemma holds:

Lemma 1.2. *Let* E *be an isometry of* $\mathbb{R}^3(0)$ *written as* $E = A + \mathbf{c}$ $(A \in SO(3), \mathbf{c} \in \mathbb{R}^3)$, *and suppose* $A \neq \mathrm{id}$. *Then there exists an isometry* P *such that*

$$P\circ E\circ P^{-1}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \tau\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.7}$$

for all $x = {}^t(x^1, x^2, x^3) \in \mathbb{R}^3(0)$. *Moreover, if* E *has such a decomposition, then the* $e^{\pm i\theta}$ *are the eigenvalues of the matrix* A *and* $\pm\tau = \langle \mathbf{c}, \mathbf{e}\rangle$, *where* $\mathbf{e}$ *is the unit eigenvector of* A *corresponding to the eigenvalue* 1 *and* $\langle\ ,\ \rangle$ *denotes the canonical inner product of* $\mathbb{R}^3$.

Proof. Let $P = P_0 + \mathbf{p}$ $(P_0 \in SO(3,1), \mathbf{p} \in \mathbb{R}^3)$. Then E has the expression $P\circ E\circ P^{-1} = R_\theta + \tau\mathbf{e}_3$ of (1.7) $(R_\theta \in SO(3,1), \mathbf{e}_3 = {}^t(0,0,1))$ if and only if

$$P_0^{-1}\circ R_\theta \circ P_0 = A, \tag{1.8}$$

$$R_\theta\mathbf{p} + \tau\mathbf{e}_3 - \mathbf{p} = P_0\mathbf{c}. \tag{1.9}$$

It is obvious that P_0 satisfying (1.8) exists. Hence, it suffices to show that the existence of $\mathbf{p}$ satisfying (1.9). Note that if such a $\mathbf{p}$ exists, then the $e^{\pm i\theta}$ are the eigenvalues of A by (1.8) and, by (1.9),

$$\tau = \langle \tau\mathbf{e}_3, \mathbf{e}_3\rangle = \langle P_0\mathbf{c}, \mathbf{e}_3\rangle = \langle \mathbf{c}, P_0^{-1}\mathbf{e}_3\rangle = \langle \mathbf{c}, \mathbf{e}\rangle.$$

Now we put $P_0\mathbf{c} = {}^t(\alpha^1, \alpha^2, \alpha^3)$. Then (1.9) is equivalent to

$$\tau = \alpha^3 \quad\text{and}\quad \begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{pmatrix}\begin{pmatrix} p^1 \\ p^2 \end{pmatrix} = \begin{pmatrix} \alpha^1 \\ \alpha^2 \end{pmatrix}.$$

Consequently, the desired τ and $\mathbf{p}$ exist if $\theta \notin 2\pi\mathbb{Z}$. □

2. The stereographic projections

Recall that

$$\mathbb{R}^3(k) = \begin{cases} \mathbb{R}^3 & (\text{if } k \geq 0), \\ \{x \in \mathbb{R}^3 : \sum_{i=1}^3 (x^i)^2 < \frac{1}{|k|}\} & (\text{if } k < 0) \end{cases}$$

is the Riemannian 3-manifold with the Riemannian metric

$$g_k = \left(\frac{2}{1 + k \sum_{i=1}^3 (x^i)^2} \right)^2 \sum_{i=1}^3 (dx^i)^2$$

of constant sectional curvature k.

Note that when $k > 0$, $\mathbb{R}^3(k)$ can be understood as the image of the stereographic projection of $M^3(k)$ defined in (1.1) into the (x^1, x^2, x^3)-plane from the south pole $(0, 0, 0, -1/\sqrt{k})$. Similarly, when $k < 0$, $\mathbb{R}^3(k)$ is also the image of the stereographic projection of $M^3(k)$ defined in (1.3) into the (x^1, x^2, x^3)-plane from the south pole $(0, 0, 0, -1/\sqrt{|k|})$.

Let ψ_k $(k \neq 0)$ denote these stereographic projections. Then ψ_k and ψ_k^{-1} are given, independently of the sign of k, by

$$\psi_k(x^1, x^2, x^3, t) = \frac{1}{\sqrt{|k|}t + 1}(x^1, x^2, x^3), \tag{2.1}$$

$$\psi_k^{-1}(x^1, x^2, x^3) = \frac{2}{1 + kr^2} \left(x^1, x^2, x^3, \frac{1 - kr^2}{2\sqrt{|k|}} \right), \tag{2.2}$$

where $r^2 = \sum_{i=1}^3 (x^i)^2$. The Riemannian metric g_k of $\mathbb{R}^3(k)$ is nothing but the one induced from the canonical metric of $M^3(k)$ by ψ_k. Therefore, isometries of $M^3(k)$ can be regarded as isometries of $\mathbb{R}^3(k)$.

Now we interpret the normalized isometries (1.2), (1.5), and (1.7) in terms of the canonical coordinate system of $\mathbb{R}^3(k)$. If $k > 0$, then a matrix $E \in SO(4)$ of the form (1.4) is expressed as

$$\begin{aligned} \psi_k \circ E \circ \psi_k^{-1} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} &= \mu \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \cos\nu \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \\ &\quad + \frac{\mu(1 - kr^2)}{2\sqrt{k}} \begin{pmatrix} 0 \\ 0 \\ -\sin\nu \end{pmatrix}, \end{aligned} \tag{2.3a}$$

where

$$\mu = 2\{ 2\sqrt{k}x^3 \sin\nu + \cos\nu(1 - kr^2) + (1 + kr^2) \}^{-1}. \tag{2.3b}$$

Note that the singular point of $\psi_k \circ E \circ \psi_k^{-1}$ corresponds to the point in $M^3(k)$ which is mapped to the south pole by E.

On the other hand, if $k < 0$, then a matrix $E \in SO^+(3, 1)$ of the form (1.5) is expressed as

$$\text{(2.4a)} \qquad \psi_k \circ E \circ \psi_k^{-1} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \mu \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \cosh\nu \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \frac{\mu(1-kr^2)}{2\sqrt{|k|}} \begin{pmatrix} 0 \\ 0 \\ \sinh\nu \end{pmatrix},$$

where

$$\text{(2.4b)} \qquad \mu = 2\{\, 2\sqrt{|k|}x^3 \sinh\nu + \cosh\nu(1-kr^2) + (1+kr^2)\,\}^{-1}.$$

In (2.3) and (2.4), we now put

$$\text{(2.5)} \qquad \tau = \begin{cases} -\frac{\sin\nu}{2\sqrt{k}} & (\text{if } k > 0), \\ \frac{\sinh\nu}{2\sqrt{|k|}} & (\text{if } k < 0), \end{cases}$$

and denote $\psi_k \circ E \circ \psi_k^{-1}$ by $T_k(\theta, \tau)$. To determine ν uniquely from τ, we assume that $|\nu| < \pi/2$ if $k > 0$. Then $T_k(\theta, \tau)$ is expressed, independently of the sign of k, by

$$\text{(2.6a)} \quad T_k(\theta, \tau) \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \tilde{\mu}_k \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & (1-4k\tau^2)^{1/2} \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \tilde{\mu}_k(1-kr^2) \begin{pmatrix} 0 \\ 0 \\ \tau \end{pmatrix},$$

where $|\tau| < 1/2\sqrt{|k|}$ for $k > 0$, and

$$\text{(2.6b)} \qquad \tilde{\mu}_k = 2\{\, -4k\tau x^3 + (1-4k\tau^2)^{1/2}(1-kr^2) + (1+kr^2)\,\}^{-1}.$$

We also define $T_k(\theta, \tau)$ and $\tilde{\mu}_k$ by (2.6a) and (2.6b) even when $k = 0$. Then $\tilde{\mu}_0 = 1$ and $T_0(\theta, \tau)$ is identical with normalized isometry given by (1.7). Thus, for each $k \in \mathbb{R}$, we call $T_k(\theta, \tau)$ a normal form of the isometry of $\mathbb{R}^3(k)$.

3. A DIFFERENTIABLE STRUCTURE OF $\mathscr{I}$

Recall that G_k is the identity component of the isometry group of $M^3(k)$, namely

$$G_k = \begin{cases} SO(4) & (k > 0), \\ SO(3) \ltimes \mathbb{R}^3 & (k = 0), \\ SO^+(3, 1) & (k < 0). \end{cases}$$

Let $\mathscr{I} = \{(k, E) : E \in G_k\}$. Then each of the subsets

$$\mathscr{I}^+ = \{(k, E) \in \mathscr{I} : k > 0\} = (0, \infty) \times SO(4),$$
$$\mathscr{I}^- = \{(k, E) \in \mathscr{I} : k < 0\} = (-\infty, 0) \times SO^+(3, 1)$$

has the canonical differentiable structures as a product. In this section we shall prove the following theorem.

Theorem 3.1. *There exists a differentiable structure on $\mathscr{I}$ whose restriction to $\mathscr{I}^+$ (resp. $\mathscr{I}^-$) is compatible with the canonical product structure of $\mathscr{I}^+$ (resp. $\mathscr{I}^-$).*

Let $\widetilde{\mathscr{I}}$ be a subset of $\mathscr{I}$ defined by

$$\widetilde{\mathscr{I}} = \mathscr{I} \setminus \{(k, E) =: k > 0 \text{ and } E \in SO(4) \text{ maps the north pole of } M^3(k) \text{ to the south pole}\}.$$

For each $(k, E) \in \widetilde{\mathscr{I}}$, we put

$$w^i(E) = \psi_k^i \circ E \circ \psi_k^{-1}(0) \qquad (i = 1, 2, 3), \tag{3.1}$$

$$w^{jl}(E) = \left[\frac{\partial}{\partial x^l}(\psi_k^j \circ E \circ \psi_k^{-1})\right](0) \qquad (j, l = 1, 2, 3), \tag{3.2}$$

and define a map $\mathscr{W} : \widetilde{\mathscr{I}} \to \mathbb{R}^{13}$ by

$$\mathscr{W}(k, E) = (k, w^i(E), w^{jl}(E))_{i,j,l=1,2,3} \in \mathbb{R}^{13},$$

where ψ_k $(k \neq 0)$ is the stereographic projection defined in §1 and ψ_0 is the identity map of $M^3(0)$. The map $\mathscr{W}$ is injective, since every isometry $E \in G_k$ is uniquely determined by the data (3.1) and (3.2). Moreover, it is easily verified that the restriction of the map $\mathscr{W}|_{\widetilde{\mathscr{I}} \cap G_k}$ is an embedding for each $k \in \mathbb{R}$.

Now we introduce some terminology. Let $U \subset \mathbb{R}^7$ be an open subset. Then an immersion $\varphi : U \to \mathbb{R}^{13}$ is said to be *admissible* if it satisfies (Image of φ) $\subset$ (Image of $\mathscr{W}$). Then we have the following

Lemma 3.2. *The image of $\mathscr{W}$ has a unique differentiable structure as an embedded submanifold of $\mathbb{R}^{13}$ such that any admissible immersion induces its local coordinate system.*

Theorem 3.1 can now follow easily from Lemma 3.2:

Proof of Theorem 3.1. Since $\mathscr{W}|_{\mathscr{I}^+}$ and $\mathscr{W}|_{\mathscr{I}^-}$ are locally admissible, the differentiable structure on $\widetilde{\mathscr{I}}$ induced by $\mathscr{W}$ is compatible with the canonical product structures of $\mathscr{I}^+$ and $\mathscr{I}^-$. Thus $\mathscr{I} = \widetilde{\mathscr{I}} \cup \mathscr{I}^+ \cup \mathscr{I}^-$ has a differentiable structure stated in the theorem with respect to the topology generated by $\{\widetilde{\mathscr{I}}, \mathscr{I}^+, \mathscr{I}^-\}$. □

Before we prove Lemma 3.2, we define the following transformations of $\mathbb{R}^3(k)$, which may have singular points when $k > 0$:

$$S_1(k, \theta, \tau)\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \mu_1 \begin{pmatrix} (1 - 4k\tau^2)^{1/2} & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \mu_1(1 - kr^2)\begin{pmatrix} \tau \\ 0 \\ 0 \end{pmatrix},$$

$$S_2(k, \theta, \tau)\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \mu_2 \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & (1-4k\tau^2)^{1/2} & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \mu_2(1-kr^2)\begin{pmatrix} 0 \\ \tau \\ 0 \end{pmatrix},$$

$$\begin{aligned} S_3(k, \theta, \tau)\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} &= T_k(\theta, \tau) \\ &= \mu_3 \begin{pmatrix} 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \\ (1-4k\tau^2)^{1/2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \mu_3(1-kr^2)\begin{pmatrix} 0 \\ \tau \\ 0 \end{pmatrix}, \end{aligned}$$

where $|\tau| < 1/2\sqrt{k}$ for $k > 0$, and

$$\begin{aligned} \mu_i &= \mu_i(k, \theta, \tau, x^1, x^2, x^3) \\ &= 2\{-4k\tau x^i + (1-4k\tau^2)^{1/2}(1-kr^2) + (1+kr^2)\}^{-1} \qquad (i = 1, 2, 3). \end{aligned}$$

By the same argument as that for $S_3(k, \theta, \tau) = T_k(\theta, \tau)$ in the previous section, we can show that $\psi_k^{-1} \circ S_i(k, \theta, \tau) \circ \psi_k \in G_k$ $(i = 1, 2)$. In fact, if $k < 0$ for instance, the corresponding three matrices in G_k are given by

$$\psi_k^{-1} \circ S_1(k, \theta, \tau) \circ \psi_k = \begin{pmatrix} \cosh\nu & 0 & 0 & \sinh\nu \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ \sinh\nu & 0 & 0 & \cosh\nu \end{pmatrix},$$

$$\psi_k^{-1} \circ S_2(k, \theta, \tau) \circ \psi_k = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cosh\nu & 0 & \sinh\nu \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sinh\nu & 0 & \cosh\nu \end{pmatrix},$$

$$\psi_k^{-1} \circ S_3(k, \theta, \tau) \circ \psi_k = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cosh\nu & \sinh\nu \\ 0 & 0 & \sinh\nu & \cosh\nu \end{pmatrix},$$

where $\tau = (\sinh\nu)/2\sqrt{|k|}$ (cf. (2.5)).

Using these, we define a smooth map $h_k : \mathbb{R}^6 \to G_k$ $(k \in \mathbb{R})$ by

$$\begin{aligned} h_k&(\theta^1, \theta^2, \theta^3, \tau^1, \tau^2, \tau^3) \\ &= \psi_k^{-1} \circ S_1(k, 0, \tau^1) \circ S_2(k, 0, \tau^2) \circ S_3(k, 0, \tau^3) \\ &\quad \circ S_1(k, \theta^1, 0) \circ S_2(k, \theta^2, 0) \circ S_3(k, \theta^3, 0) \circ \psi_k. \end{aligned}$$

Then one can easily verify that h_k defines locally a diffeomorphism from a neighborhood of the origin onto a neighborhood of the identity.

Proof of Lemma 3.2. By the implicit function theorem, it is sufficient to show that for each $(k, E) \in \widetilde{\mathscr{F}}$, there exists an admissible immersion $\varphi : U \subset \mathbb{R}^7 \to$

$\mathbb{R}^{13}$ such that $(k, E) \in$ (Image of φ). Since $\mathscr{W}|_{\mathscr{I}^+}$ and $\mathscr{W}|_{\mathscr{I}^-}$ are locally admissible, the existence of such a φ is obvious for $(k, E) \in \widetilde{\mathscr{I}}$ $(k \neq 0)$. Now let $(0, E) \in \widetilde{\mathscr{I}}$. Then, since $h_0 : \mathbb{R}^6 \to G_0$ is surjective, there exists a point $\mathbf{a} \in \mathbb{R}^6$ such that $E \in h_0(\mathbf{a})$. We define a smooth map $\varphi : \mathbb{R}^7 \to \mathbb{R}^{13}$ by

$$\varphi(k, \theta^1, \theta^2, \theta^3, \tau^1, \tau^2, \tau^3) = \mathscr{W}(h_k(\mathbf{a}) \circ h_k(\theta^1, \theta^2, \theta^3, \tau^1, \tau^2, \tau^3)).$$

Since $\mathscr{W}|_{G_0}$ is an immersion and h_0 is nonsingular at the origin, it is easy to see that the rank of φ at the origin is 7. (We need not calculate the derivative of φ with respect to k because both the domain and the range of φ have the same parameter k.) Thus φ defines an admissible immersion on some neighborhood of the origin such that $(0, E) \in$ (Image of φ).

These results can be extended to a higher-dimensional case. In fact, let $M^n(k)$ be a complete simply connected Riemannian n-manifold of constant sectional curvature k, and $G_k^{(n)}$ the identity component of its isometry group. Then by the same argument as above a differentiable structure on $\mathscr{I}^{(n)} = \{(k, E) : E \in G_k^{(n)}, k \in \mathbb{R}\}$ can also be introduced. Furthermore, Tasaki-Umehara-Yamada [3] developed these results for symmetric spaces. We apply these results to hypersurfaces in $M^n(k)$ as follows. Let M be a compact hypersurface of $M^n(k)$. Then the induced metric g and the second fundamental form h satisfy the Gauss and Codazzi equations:

$$(\mathbf{Ga}_k)\quad \begin{aligned} R(X, Y, Z, W) = k\{&g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\} \\ &+ h(X, Z)h(Y, W) - h(X, W)h(Y, Z) \\ &\qquad\qquad (X, Y, Z, W \in TM), \end{aligned}$$

$$(\mathbf{Co})\qquad (\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z) \qquad (X, Y, Z \in TM),$$

where ∇ is the Levi-Civita connection of g and R denotes its curvature tensor. Now, let g_k $(k \in \mathbb{R})$ be a smooth one-parameter family of Riemannian metrics on a compact $(n-1)$-manifold M and h_k $(k \in \mathbb{R})$ a smooth one-parameter family of symmetric 2-tensors such that g_k and h_k satisfy $(\mathbf{Ga}_k)$ and $(\mathbf{Co})$ for each k. It then follows from the fundamental theorem for hypersurfaces that there exists an immersion $f_k : \widetilde{M} \to M^n(k)$ whose induced metric and second fundamental form coincide with $\pi^* g_k$ and $\pi^* h_k$ respectively, where $\pi : \widetilde{M} \to M$ is the universal covering of M. Note that each deck transformation T of $\widetilde{M}$ preserves $\pi^* g_k$ and $\pi^* h_k$ and hence T extends to an isometry of $M^n(k)$ by the rigidity of f_k. Thus, for each k, we have a representation $\rho_k : \pi_1(M) \to G_k^{(n)}$. Then the following holds.

Proposition 3.3. *The family of the representation*

$$\rho_k : \pi_1(M) \to G_k^{(n)} \subset \mathscr{I}^{(n)} \qquad (k \in \mathbb{R})$$

depends smoothly on the parameter k with respect to the differentiable structure of $\mathscr{I}^{(n)}$.

Proof. We confine our discussion to the case $n = 3$. But the following proof is valid also for the higher-dimensional case. Let $p \in M$ and choose a reference point $q_0 \in \pi^{-1}(p)$. Then each deck transformation T determines uniquely a point $q \in \pi^{-1}(p)$ such that $T(q_0) = q$. If we normalize $\psi_k \circ f_k(q_0) = 0$ and take a frame (e_1, e_2) of (M, g_k) at p, then the isometry E_k corresponding to T satisfies

$$\text{(3.3)} \qquad \begin{gathered} \widetilde{E}_k(0) = \psi_k \circ f_k(q), \qquad d\widetilde{E}_k(\xi_{q_0}) = \xi_q, \\ d(\widetilde{E}_k \circ \psi_k \circ f_k)[(d\pi^{-1})_{q_0}(e_j)] = d(\psi_k \circ f_k)[(d\pi^{-1})_q(e_j)] \quad (j = 1, 2), \end{gathered}$$

where $\widetilde{E}_k = \psi_k \circ E_k \circ \psi_k^{-1} : \mathbb{R}^3(k) \to \mathbb{R}^3(k)$ and ξ is the unit normal vector field of f_0. Since the coefficients of the Frenet equation with respect to the canonical coordinate system of $\mathbb{R}^3(k)$ depends smoothly on k, so does $\psi_k \circ f_k : M \to \mathbb{R}^3(k)$. Thus (3.3) implies that $\mathscr{W}(E_k) \in \mathbb{R}^{13}$ is smooth with respect to k. So, by the definition of our differentiable structure of $\mathscr{I}$, E_k depends smoothly on k. $\square$

4. Smoothness of Normal form

Let

$$\begin{aligned} \mathscr{N}^- &= \{(k, E) \in \mathscr{I} : k < 0 \text{ and all the eigenvalues} \\ &\qquad\qquad\qquad\qquad\qquad \text{of } E \in SO^+(3, 1) \text{ are } 1\}, \\ \mathscr{N}^0 &= \{(0, E) \in \mathscr{I} : E \text{ is the identity or a translation of } \mathbb{R}^3\}, \\ \mathscr{N}^+ &= \{(k, E) \in \mathscr{I} : k > 0 \text{ and } \theta \geq \nu \text{ or } \cos\nu \leq 0 \\ &\qquad\qquad \text{in the decomposition (1.2) of } E \in SO(4)\}, \end{aligned}$$

and define a closed subset in $\mathscr{I}$ by

$$\mathscr{N} = \mathscr{N}^- \cup \mathscr{N}^0 \cup \mathscr{N}^+. \tag{4.1}$$

Then for each $(k, E) \in \widetilde{\mathscr{I}} \backslash \mathscr{N}$ there exists $(k, P) \in \mathscr{I}$ such that

$$P^{-1} \circ E \circ P = T_k(\theta, \tau), \tag{4.2}$$

where $T_k(\theta, \tau)$ is the normal form defined in §2. In §1 it was proved that θ and τ are determined up to $\mathbb{Z}_2$-ambiguity. In this section, we will see that locally θ and τ are smooth functions on $\widetilde{\mathscr{I}} \backslash \mathscr{N}$ with respect to the differentiable structure defined in §3.

Theorem 4.1. *Let $(k, E) \in \widetilde{\mathscr{I}} \backslash \mathscr{N}$. Then there exists a neighborhood $U \subset \widetilde{\mathscr{I}} \backslash \mathscr{N}$ of (k, E) such that, by taking suitable branches, θ and τ in (4.2) are defined as smooth functions on U.*

If $k \neq 0$, the theorem is obvious. So we may assume $k = 0$. Since each $E \in G_0 \backslash \mathscr{N}^0$ is equivalent to a normal form $T_0(\alpha, \beta)$ $(\alpha, \beta \in \mathbb{R}, \alpha \notin 2\pi\mathbb{Z})$ by (4.2), it is sufficient to prove the following lemma.

Lemma 4.2. *Let $\mathscr{U} : \mathbb{R}^7 \to \mathscr{I}$ be a map defined by*

$$\begin{aligned} \mathscr{U}(k, \theta^1, \theta^2, \theta^3, \tau^1, \tau^2, \tau^3) = S_1(k, \theta^1, \tau^1) \circ S_2(k, \theta^2, \tau^2) \circ T_k(\theta^3, \tau^3) \\ \circ S_2^{-1}(k, \theta^2, \tau^2) \circ S_1^{-1}(k, \theta^1, \tau^1). \end{aligned}$$

Then the Jacobian of $\mathscr{U}$ does not vanish at the point $(0, 0, 0, \alpha, 0, 0, \beta)$ $(\alpha \notin 2\pi\mathbb{Z})$.

Proof. Consider the map $\mathscr{W} \circ \mathscr{U} : \mathbb{R}^7 \to \mathbb{R}^{13}$ Then

$$\operatorname{rank}(d(\mathscr{W} \circ \mathscr{U})) = \operatorname{rank} \frac{\partial(k, w^i, w^{jl})}{\partial(k, \theta^1, \theta^2, \theta^3, \tau^1, \tau^2, \tau^3)} = 1 + \operatorname{rank} \left.\frac{\partial(w^i, w^{jl})}{\partial(\theta^1, \theta^2, \theta^3, \tau^1, \tau^2, \tau^3)}\right|_{k=0}$$

at the point $(0, 0, 0, \alpha, 0, 0, \beta)$. By a straightforward calculation, the derivatives on the right-hand side are given by

$$\begin{aligned}
dw^1 &= (0, -\beta, 0, 0, 1-\cos\alpha, \sin\alpha),\\
dw^2 &= (-\beta, 0, 0, 0, -\sin\alpha, 1-\cos\alpha),\\
dw^3 &= (\beta, \beta, 0, 2\beta, 2\beta, 1),\\
dw^{11} &= (0, 0, -\sin\alpha, 2\cos\alpha, 2\cos\alpha, 0),\\
dw^{21} &= (0, 0, \cos\alpha, 2\sin\alpha, 2\sin\alpha, 0),\\
dw^{13} &= (-\sin\alpha, -1+\cos\alpha, 0, 0, 0, 0),\\
dw^{23} &= (-1+\cos\alpha, -\sin\alpha, 0, 0, 0, 0),
\end{aligned}$$

which yield

$$\det\left\{\frac{\partial(w^1, w^2, w^3, w^{11}, w^{13}, w^{23})}{\partial(\theta^1, \theta^2, \theta^3, \tau^1, \tau^2, \tau^3)}\right\} = 8\sin\alpha(1-\cos\alpha)^2,$$

$$\det\left\{\frac{\partial(w^1, w^2, w^3, w^{21}, w^{13}, w^{23})}{\partial(\theta^1, \theta^2, \theta^3, \tau^1, \tau^2, \tau^3)}\right\} = -8\cos\alpha(1-\cos\alpha)^2.$$

Hence, $d\mathscr{U}$ is nondegenerate at $(0, 0, 0, \alpha, 0, 0, \beta)$, since $\alpha \notin 2\pi\mathbb{Z}$. □

By Theorem 4.1, we may regard θ and τ as globally defined functions $\theta : \widetilde{\mathscr{I}\backslash\mathscr{N}} \to \mathbb{R}/2\pi\mathbb{Z}$ and $\tau : \widetilde{\mathscr{I}\backslash\mathscr{N}} \to \mathbb{R}$.

5. Deformation of the immersion

Let $\Omega(a_0, b_0) = (-a_0, a_0)\times(-b_0, b_0)$ be a rectangular domain of $\mathbb{R}^2$. Then the Dirichlet problem of the sinh-Gordon equation

$$\Delta\omega + \cosh\omega \sinh\omega = 0 \tag{5.1}$$

on $\Omega(a_0, b_0)$ has a unique positive solution ω_0 if $a_0^{-2} + b_0^{-2} > 4\pi^{-2}$ [7, 1, 2]. By the odd reflections about $\partial\Omega(a_0, b_0)$, this solution can be extended to a doubly periodic solution $\widetilde{\omega}_0$ of (5.1), which has a rectangular fundamental domain. To get solutions with twisted fundamental domain, we can perturb $\widetilde{\omega}_0$ in the following fashion.

Lemma 5.1 [6, Theorem 1]. *For sufficiently small $a_0, b_0 > 0$, there exist a neighborhood U of $(a_0, b_0, 0) \in \mathbb{R}^3$ and a smooth function $\omega(u, v; a, b, c)$ on $\mathbb{R}^2 \times U$ which satisfy the following conditions:*

(1) *For each $(a, b, c) \in U$, $\omega(u, v; a, b, c)$ is a solution of* (5.1) *on $\mathbb{R}^2$.*

(2) *Let $\mathbf{p}_1 = (2a, 0)$ and $\mathbf{p}_2 = (2c, 2b)$. Then*

$$\omega(\mathbf{u}+\mathbf{p}_1; \mathbf{a}) = \omega(\mathbf{u}+\mathbf{p}_2; \mathbf{a}) = \omega(-\mathbf{u}; \mathbf{a}) = -\omega(\mathbf{u}; \mathbf{a}), \tag{5.2}$$

where $\mathbf{u} = (u, v)$ and $\mathbf{a} = (a, b, c)$.

(3) $\omega(u, v; a_0, b_0, 0) = \widetilde{\omega}_0(u, v)$.

Let $\omega(u, v) = \omega(u, v; a, b, c)$ be a doubly periodic solution determined as above. Define the first fundamental form ds^2 by

$$ds^2 = \frac{e^{2\omega}}{4(H^2+k)}(du^2+dv^2), \tag{5.3}$$

and the second fundamental form $h = h_{11}du^2 + 2h_{12}du\,dv + h_{22}dv^2$ by

$$\begin{aligned} h_{11} &= \frac{He^{2\omega}}{4(H^2+k)} - \frac{\cos 2\beta}{4(H^2+k)^{1/2}}, \\ h_{12} &= -\frac{\sin 2\beta}{4(H^2+k)^{1/2}}, \\ h_{22} &= \frac{He^{2\omega}}{4(H^2+k)} + \frac{\cos 2\beta}{4(H^2+k)^{1/2}}. \end{aligned} \tag{5.4}$$

Then it is not hard to see that for $H \equiv 1/2$ and $k > -1/4$, ds^2 and h satisfy the Gauss and Codazzi equations in $\mathbb{R}^3(k)$. Hence, by the fundamental theorem for surfaces, they determine, up to an isometry of $\mathbb{R}^3(k)$, an isometric immersion $f_k = f_k(a, b, c, \beta)\colon (\mathbb{R}^2, ds^2) \to \mathbb{R}^3(k)$ with constant mean curvature $H \equiv 1/2$. Since the Frenet equation of f_k with respect to the canonical coordinates on $\mathbb{R}^3(k)$ depends smoothly on k, the immersion $f_k(a, b, c, \beta) : \mathbb{R}^2 \to \mathbb{R}^3(k)$ also depends smoothly on the parameters a, b, c, β, and k.

Since ω has the doubly periodicity condition (5.2), there exist motions $E_i = E_i(k, a, b, c, \beta)$ $(i = 1, 2)$ of $\mathbb{R}^3(k)$ such that

$$f_k(\mathbf{u} + 2\mathbf{p}_i;\ a, b, c, \beta) = E_i \circ f_k(\mathbf{u};\ a, b, c, \beta) \qquad (i = 1, 2). \tag{5.5}$$

It follows from Proposition 3.3 that $E_i(k, a, b, c, \beta)$ $(i = 1, 2)$ are smooth with respect to the parameters a, b, c, β, and k.

Properties of the immersions $f_k = f_k(a, b, c, \beta)$ at $k = 0$ are carefully analyzed by Wente [8], in which those of the form $f_0(a, b, 0, 0)$ $(a^{-2}+b^{-2} > 4\pi^{-2})$ whose images are compact are called *symmetric examples.* The existence of symmetric examples has been shown in Wente [7], Abresch [1], and Walter [5]. Now we assume that $f_0(a_0, b_0, 0, 0)$ yields a symmetric example. Then we may put

$$E_1(0, a_0, b_0, 0, 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$E_2(0, a_0, b_0, 0, 0) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\pi < \alpha < 2\pi$ and $\alpha \in 2\pi\mathbb{Q}$ (see [1, 8]).

Note that, since $E_1(0, a_0, b_0, 0, 0) \in \mathcal{N}$, Theorem 4.1 cannot apply directly. So we change a generator $\mathbf{p}_1$ of the lattice $\Gamma = \{\mathbf{p}_1, \mathbf{p}_2\}$ for

$$\mathbf{p}_3 = \mathbf{p}_1 + \mathbf{p}_2.$$

Let

$$E_3(k, a, b, c, \beta) = E_1(k, a, b, c, \beta) \circ E_2(k, a, b, c, \beta).$$

Then it is obvious that

$$f_k(\mathbf{u}+2\mathbf{p}_3;\ a, b, c, \beta) = E_3 \circ f_k(\mathbf{u};\ a, b, c, \beta). \tag{5.6}$$

Now we prove our main result:

Theorem 5.2. *Let T^2 be a compact 2-manifold with genus 1. Then for sufficiently small $\varepsilon > 0$, there exists a 1-parameter family of immersions $f_k : T^2 \to \mathbb{R}^3(k)$ $(|k| < \varepsilon)$ with constant mean curvature $H \equiv 1/2$.*

Proof. Using Theorem 4.1, we can define smooth functions $\tilde{\theta}_i$ and $\tilde{\tau}_i$ $(i = 2, 3)$ on some neighborhood of $(0, a_0, b_0, 0, 0) \in \mathbb{R}^5$ by

$$\begin{aligned} \tilde{\theta}_i(k, a, b, c, \beta) &= \theta(E_i(k, a, b, c, \beta)) \\ \tilde{\tau}_i(k, a, b, c, \beta) &= \tau(E_i(k, a, b, c, \beta)) \end{aligned} \qquad (i = 2, 3).$$

Thus, to prove the theorem, it suffices to show that the set

$$\{(k, a, b, c, \beta) \in U : \tilde{\theta}_i(k, a, b, c, \beta) \equiv \alpha \in 2\pi\mathbb{Q} \\ \text{and } \tilde{\tau}_i(k, a, b, c, \beta) = 0\ (i = 2, 3)\}$$

defines a regular curve with respect to k through the point $(0, a_0, b_0, 0, 0)$. To see this, we define a map $\varphi : U \to \mathbb{R}^5$ by

$$\varphi(k, a, b, c, \beta) = (k, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\tau}_2, \tilde{\tau}_3).$$

In [8] Wente introduced functions θ_1, θ_2, τ_1, and τ_2 of variables a, b, c, and β in such a way that $E_i(a, b, c, \beta)$ $(i = 1, 2)$ are equivalent to $T_0(\theta_i, \tau_i)$, for which he showed that

$$\det\left\{\frac{\partial(\theta_1, \theta_2, \tau_1, \tau_2)}{\partial(a, b, c, \beta)}\right\} \neq 0 \tag{5.7}$$

at $(a_0, b_0, 0, 0)$. It is easily verified that these functions are related to $\tilde{\theta}_2$, $\tilde{\theta}_3$, $\tilde{\tau}_2$ and $\tilde{\tau}_3$ by

$$\begin{aligned} \tilde{\theta}_2(0, a, b, c, \beta) &= \theta_2(a, b, c, \beta), \\ \tilde{\theta}_3(0, a, b, c, \beta) &= \theta_1(a, b, c, \beta) + \theta_2(a, b, c, \beta), \\ \tilde{\tau}_2(0, a, b, c, \beta) &= \tau_2(a, b, c, \beta), \\ \tilde{\tau}_3(0, a, b, c, \beta) &= \tau_1(a, b, c, \beta) + \tau_2(a, b, c, \beta). \end{aligned}$$

Hence we have from (5.7)

$$\begin{aligned} \operatorname{rank}(d\varphi) &= \operatorname{rank}\left\{\frac{\partial(k, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\tau}_2, \tilde{\tau}_3)}{\partial(k, a, b, c, \beta)}\right\} \\ &= 1 + \operatorname{rank}\left\{\left.\frac{\partial(\tilde{\theta}_2, \tilde{\theta}_3, \tilde{\tau}_2, \tilde{\tau}_3)}{\partial(a, b, c, \beta)}\right|_{k=0}\right\} \\ &= 1 + \operatorname{rank}\left\{\frac{\partial(\theta_1, \theta_2, \tau_1, \tau_2)}{\partial(a, b, c, \beta)}\right\} = 5 \end{aligned}$$

at $(0, a_0, b_0, 0, 0)$. Thus $\varphi^{-1}(k, \alpha, 0, \alpha, 0)$ determines a regular curve on some small neighborhood of $(0, a_0, b_0, 0, 0) \in U$. $\square$

Corollary 5.3. *Any open subset of the 3-sphere or the hyperbolic 3-space contains a torus with constant mean curvature.*

Proof. Let $\{f_k : T^2 \to \mathbb{R}^3(k)\}_{|k|<\varepsilon}$ be as in Theorem 5.2. Then, for sufficiently small ε, the images $\{f_k(T^2)\}_{|k|<\varepsilon}$ are uniformly bounded. Namely, $f_k(T^2)$ is contained in the ball of radius a with respect to g_k for each $k \in (-\varepsilon, \varepsilon)$, where $a > 0$ is a universal constant.

Assume $k < 0$ and define

$$\tilde{f} = \sqrt{|k|} \cdot (\psi_k^{-1} \circ f_k) : T^2 \to M^3(-1),$$

where ψ_k is the stereographic projection (2.1) and $\cdot$ is the scalar multiplication in $M^3(k) \subset \mathbb{L}^4$. Then $\tilde{f}$ gives an immersion of T^2 into the hyperbolic 3-space $M^3(-1)$ with constant mean curvature $1/2\sqrt{|k|}$. Moreover, $\tilde{f}(T^2)$ is contained in the ball of radius $\sqrt{|k|}\,a$ in $M^3(-1)$, since $f_k(T^2)$ is bounded by the ball with radius a.

Hence, taking a sufficiently small $k < 0$, we can find an immersion of T^2 with constant mean curvature into the hyperbolic 3-space with sufficiently small radius.

Similarly, assuming $k > 0$, we have the conclusion for the 3-sphere. $\square$

By using (5.7), the existence of nonholomorphic harmonic maps of tori generated by any lattice into the unit sphere has been shown in Umehara-Yamada [4], which is based on the fact that Gauss maps of surfaces with constant mean curvature in $\mathbb{R}^3(0)$ are harmonic.

References

1. U. Abresch, *Constant mean curvature tori in terms of elliptic functions*, J. Reine Angew. Math. **374** (1987), 169–192.
2. J. Spruck, *The elliptic sinh-Gordon equation and the construction of toroidal soap bubbles*, Lecture Notes in Math., vol.1340, Springer-Verlag, 1988, pp. 275–381.
3. H. Tasaki, M. Umehara, and K. Yamada, *Deformations of symmetric spaces of compact type to their noncompact duals*, Japan. J. Math. **17** (1991) (to appear).
4. M. Umehara and K. Yamada, *Harmonic non-holomorphic maps of 2-tori into the 2-sphere*, Geometry of Manifolds, Academic Press, 1989, pp. 151–161.
5. R. Walter, *Explicit examples to the H-problem of Heinz Hopf*, Geom. Dedicata **23** (1987), 187–213.
6. ——, *Constant mean curvature tori with spherical curvature lines in noneuclidean geometry*, Manuscripta Math. **63** (1989), 343–363.
7. H. C. Wente, *Counterexamples to a conjecture of H. Hopf*, Pacific J. Math. **121** (1986), 193–243.
8. ——, *Twisted tori in constant mean curvature in* $\mathbf{R}^3$, Seminar on New Results in Nonlinear Partial Differential Equations, Max-Plank-Institut für Mathematik, Bonn, 1987, pp. 1–36.

Institute of Mathematics, University of Tsukuba, Tsukuba Ibaraki 305, Japan

Faculty of General Education, Kumamoto University, Kumamoto 860, Japan